
ON THE ENERGY EXCHANGE BETWEEN RESONANT MODES IN NONLINEAR SCHRÖDINGER EQUATIONS

by

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Abstract. — We consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} \pm 2 \cos 2x |\psi|^2 \psi, \quad x \in S^1, \quad t \in \mathbb{R}$$

and we prove that the solution of this equation, with small initial datum $\psi_0 = \varepsilon(\cos x + \sin x)$, will periodically exchange energy between the Fourier modes e^{ix} and e^{-ix} . This beating effect is described up to time of order $\varepsilon^{-9/4}$ while the frequency is of order ε^2 . We also discuss some generalizations.

Résumé (Echange d'énergie entre modes résonants dans une équation de Schrödinger non linéaire cubique.)

Nous considérons l'équation de Schrödinger non linéaire

$$i\psi_t = -\psi_{xx} \pm 2 \cos 2x |\psi|^2 \psi, \quad x \in S^1, \quad t \in \mathbb{R}$$

et nous montrons la solution de cette équation ayant pour donnée initiale $\psi_0 = \varepsilon(\cos x + \sin x)$ avec ε petit, va échanger périodiquement de l'énergie entre les modes de Fourier e^{ix} et e^{-ix} . Cet effet de battement, dont la période est de l'ordre de ε^{-2} , est mis en évidence pour des temps de l'ordre de $\varepsilon^{-9/4}$. Nous présentons aussi quelques généralisations.

1. Introduction

Let us consider the non-linear Schrödinger equation (NLS) on the circle S^1

$$(1.1) \quad i\psi_t = -\psi_{xx} + V * \psi + g(x, \psi, \bar{\psi})$$

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where $V * \Psi$ denotes the convolution function between a potential $V : S^1 \mapsto \mathbb{R}$ and the function Ψ . The nonlinear term g is Hamiltonian in the sense that $g = \partial_{\bar{\psi}} G$ with G analytic with respect to its three variables and $G(x, z, \bar{z}) \in \mathbb{R}$. We further assume that G is globally at least of order three in (z, \bar{z}) at the origin in such a way g is effectively a nonlinear term.

The Fourier basis $\exp(ijx)$, $j \in \mathbb{Z}$ provides an orthonormal basis for $L^2(S^1)$ in which the linear operator $A = -\frac{\partial^2}{\partial x^2} + V*$ is diagonal. The corresponding eigenvalues of A are given by the real numbers $\omega_j = j^2 + \hat{V}(j)$, $j \in \mathbb{Z}$, where $\hat{V}(j)$ are the Fourier coefficients of V :

$$(1.2) \quad \hat{V}(j) = \int_{-\pi}^{\pi} V(x) \exp(-ijx) dx$$

where dx denotes the normalized Lebesgue measure on S^1 .

Under a non-resonant condition on the frequencies ω_j , it has been established by D. Bambusi, B. Grebert [BG06] (see also [Gré07]) that, given a number $r \geq 3$, for small initial data in H^s norm⁽¹⁾, say $\|\psi_0\|_s = \varepsilon$ with s large enough, the solution to the NLS equation (1.1) remains small in the same Sobolev norm, $\|\psi(t)\|_s \leq 2\varepsilon$ for a large period of time, $|t| \leq \varepsilon^{-r}$. Furthermore the actions $I_j \equiv \xi_j \eta_j$, $j \in \mathbb{Z}$, are almost invariant during the same period of time. In [BG06], it is also proved that in the more natural case of multiplicative potential,

$$i\psi_t = -\psi_{xx} + V\psi + g(x, \psi, \bar{\psi}),$$

which corresponds to an asymptotically resonant case, $\omega_j \sim \omega_{-j}$ when $|j| \rightarrow \infty$, we can generically impose non resonant conditions on the frequencies $\{\omega_{|j|}\}$ in such a way that the generalized actions $J_j = I_j + I_{-j}$ are almost conserved quantities. A priori nothing prevent I_j and I_{-j} from interacting. In this article, we exhibit nonlinearity g that incites this interaction and especially a beating effect. This is a nonlinear effect which is a consequence of the resonances of the linear part. We will see that not all nonlinearities incite the beating. For instance if g does not depend on x there is no beating for a long time. Typically, to obtain an interaction between the mode j and the mode $-j$, the nonlinearity g must contain oscillations of frequency $2j$ or a multiple of $2j$, i.e. g must depend on x as $\cos 2k j x$ or $\sin 2k j x$ for some $k \geq 1$. In what follows, we will focus on the totally resonant case $V = 0$ and a cubic nonlinearity $g = 2 \cos 2x |\psi|^2 \psi$. Precisely we consider the following Cauchy problem

$$(1.3) \quad \begin{cases} i\psi_t &= -\psi_{xx} \pm 2 \cos 2x |\psi|^2 \psi, & x \in S^1, t \in \mathbb{R} \\ \psi(0, x) &= \varepsilon(\cos x + \sin x) \end{cases}$$

1. Here H^s denotes the standard Sobolev Hilbert space on S^1 and $\|\cdot\|_s$ its associated norm.

where ε is a small parameter. By classical arguments based on the conservation of the energy and Sobolev embeddings, this problem has a unique global solution in all the Sobolev spaces H^s for $s \geq 1$ and even for $s \geq 0$ using more refined technics (see for instance [Bou99] chap. 5). Our result precises the behavior of the solution for time of order $\varepsilon^{-9/4}$. We write the solution ψ in Fourier modes: $\psi(t, x) = \sum_{k \in \mathbb{Z}} \hat{\psi}_k(t) e^{ikx}$.

Theorem 1.1. — *For ε small enough we have for $|t| \leq \varepsilon^{-9/4}$*

$$(1.4) \quad \begin{cases} |\hat{\psi}_1(t)|^2 + |\hat{\psi}_{-1}(t)|^2 = \varepsilon^2 + O(\varepsilon^3), \\ |\hat{\psi}_1(t)|^2 - |\hat{\psi}_{-1}(t)|^2 = \pm \varepsilon^2 \sin 2\varepsilon^2 t + O(\varepsilon^9/4). \end{cases}$$

In other words, the first estimate says that all the energy remains concentrated on the two Fourier modes $+1$ and -1 and the second estimate says that there is energy exchange, namely a beating effect, between these two modes. Notice that the sign in front of the nonlinearity does not affect the phenomena. That the principal term of g be cubic is certainly not necessary but convenient for calculus. We will see (see section 4) that the period of the beating depends on the nonlinearity but also on the value of all the initial actions.

The paper is organized as follows: in section 2 we prove a normal form result for the equation (1.3) that allows to reduce the initial problem to the study of a finite dimensional classical system. The main theorem is proved in section 3. Section 4 is devoted to generalizations and comments.

2. The normal form

Let us expand ψ and $\bar{\psi}$ in Fourier modes:

$$\psi(x) = \sum_{j \in \mathbb{Z}} \xi_j e^{ijx}, \quad \bar{\psi}(x) = \sum_{j \in \mathbb{Z}} \eta_j e^{-ijx}.$$

In this Fourier setting the equation (1.3) reads as an infinite Hamiltonian system

$$(2.1) \quad \begin{cases} i\dot{\xi}_j = j^2 \xi_j + \frac{\partial P}{\partial \eta_j} & j \in \mathbb{Z}, \\ -i\dot{\eta}_j = j^2 \eta_j + \frac{\partial P}{\partial \xi_j} & j \in \mathbb{Z} \end{cases}$$

where the perturbation term is given by

$$(2.2) \quad \begin{aligned} P(\xi, \eta) &= \pm 2 \int_{S^1} \cos 2x |\psi(x)|^4 dx \\ &= \pm \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j, \ell) = \pm 2}} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}. \end{aligned}$$

and $\mathcal{M}(j, \ell) = j_1 + j_2 - \ell_1 - \ell_2$ denotes the momentum of the multi-index $(j, \ell) \in \mathbb{Z}^4$. In the sequel we will develop the calculus with $\pm = +$ but all remain true, mutadis mutandi, with the minus sign.

Since the regularity is not an issue in this work, we will work in the phase space $(\rho \geq 0)$

$$\mathcal{A}_\rho = \{(\xi, \eta) \in \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \mid \|(\xi, \eta)\|_\rho := \sum_{j \in \mathbb{Z}} e^{\rho|j|}(|\xi_j| + |\eta_j|) < \infty\}$$

that we endow with the canonical symplectic structure $-i \sum_j d\xi_j \wedge \eta_j$. Notice that this Fourier space corresponds to functions $\psi(z)$ analytic on a strip $|\operatorname{Im} z| < \rho$ around the real axis.

According to this symplectic structure, the Poisson bracket between two functions f and g of (ξ, η) is defined by

$$\{f, g\} = -i \sum_{j \in \mathbb{Z}} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} - \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j}.$$

In particular, if $(\xi(t), \eta(t))$ is a solution of (2.1) and F is some regular Hamiltonian function, we have

$$\frac{d}{dt} F(\xi(t), \eta(t)) = \{F, H\}(\xi(t), \eta(t))$$

where $H = H_0 + P = \sum_{j \in \mathbb{Z}} j^2 \xi_j \eta_j + P$ is the total Hamiltonian of the system. We denote by $B_\rho(r)$ the ball of radius r centered at the origin in \mathcal{A}_ρ . In the next proposition we put the Hamiltonian H in normal form up to order 4:

Proposition 2.1. — *There exists a canonical change of variable τ from $B_\rho(\varepsilon)$ into $B_\rho(2\varepsilon)$ with ε small enough such that*

$$(2.3) \quad H \circ \tau = H_0 + Z_{4,1} + Z_{4,2} + Z_{4,3} + R_5$$

where

- (i) $H_0(\xi, \eta) = \sum_{j \in \mathbb{Z}} j^2 \xi_j \eta_j$,
- (ii) $Z_4 = Z_{4,1} + Z_{4,2} + Z_{4,3}$ is the (resonant) normal form at order 4, i.e. Z_4 is a polynomial of order 4 satisfying $\{H_0, Z_4\} = 0$.
- (iii) $Z_{4,1}(\xi, \eta) = 2(\xi_1 \eta_{-1} + \xi_{-1} \eta_1) \left(2 \sum_{p \in \mathbb{Z}, p \neq 1, -1} \xi_p \eta_p + (\xi_1 \eta_1 + \xi_{-1} \eta_{-1}) \right)$ is the effective hamiltonian at order 4.
- (vi) $Z_{4,2}(\xi, \eta) = 4(\xi_2 \xi_{-1} \eta_{-2} \eta_1 + \xi_{-2} \xi_1 \eta_2 \eta_{-1})$.
- (v) $Z_{4,3}$ contains all the term in Z_4 involving at most one mode of index 1 or -1.
- (vi) R_5 is the remainder of order 5, i.e. a hamiltonian satisfying $\|X_{R_5}(z)\|_\rho \leq C\|z\|_\rho^4$ for $z = (\xi, \eta) \in B_\rho(\varepsilon)$.
- (vii) τ is close to the identity: there exist a constant C_ρ such that $\|\tau(z) - z\|_\rho \leq C_\rho \|z\|_\rho^2$ for all $z \in B_\rho(\varepsilon)$.

Proof. — The proof uses the classical Birkhoff normal form procedure (see for instance [Mos68] in a finite dimensional setting or [Gré07] in the infinite dimensional ones). Since the free frequencies are the square of the integers, we are in a totally resonant case and we are not facing the small denominator problems: a linear combination of integer with integer coefficients is exactly 0 or its modulus equals at least 1. For convenience of the reader, we briefly recall the procedure. Let us search τ as time one flow of χ a polynomial Hamiltonian of order 4,

$$(2.4) \quad \chi = \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j, \ell) = \pm 2}} a_{j, \ell} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}.$$

Let F be a Hamiltonian, one has by using the Taylor expansion of $F \circ \Phi_\chi^t$ between $t = 0$ and $t = 1$:

$$F \circ \tau = F + \{F, \chi\} + \frac{1}{2} \int_0^1 (1-t) \{\{F, \chi\}, \chi\} \circ \Phi_\chi^t dt.$$

Applying this formula to $H = H_0 + P$ we get

$$H \circ \tau = H_0 + P + \{H_0, \chi\} + \{P, \chi\} + \frac{1}{2} \int_0^1 (1-t) \{\{H, \chi\}, \chi\} \circ \Phi_\chi^t dt.$$

Therefore in order to obtain $H \circ \tau = H_0 + Z_4 + R_5$ we need to solve the homological equation

$$(2.5) \quad \{\chi, H_0\} + Z_4 = P$$

and then we define

$$(2.6) \quad R_5 = \{P, \chi\} + \frac{1}{2} \int_0^1 (1-t) \{\{H, \chi\}, \chi\} \circ \Phi_\chi^t dt.$$

For $j, \ell \in \mathbb{Z}^2$ we define the associated divisor by

$$\Omega(j, \ell) = j_1^2 + j_2^2 - \ell_1^2 - \ell_2^2.$$

The homological equation 2.5 is solved by defining

$$(2.7) \quad \chi = \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j, \ell) = \pm 2, \Omega(j, \ell) \neq 0}} \frac{1}{i\Omega(j, \ell)} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}$$

and

$$(2.8) \quad Z_4 = \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j, \ell) = \pm 2, \Omega(j, \ell) = 0}} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}$$

At this stage we remark that any formal polynomial

$$Q = \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j, \ell) = \pm 2}} a_{j, \ell} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}$$

is well defined and continuous (and thus analytic) on \mathcal{A}_ρ as soon as the $a_{j\ell}$ form a bounded family. Namely, if $|a_{j, \ell}| \leq M$ for all $j, \ell \in \mathbb{Z}^2$ then

$$|Q(\xi, \eta)| \leq M \|(\xi, \eta)\|_0^4 \leq M \|(\xi, \eta)\|_\rho^4.$$

Furthermore the associated vector field is bounded (and thus smooth) from \mathcal{A}_ρ to \mathcal{A}_ρ , namely

(2.9)

$$\begin{aligned} \|X_Q(\xi, \eta)\|_\rho &= \sum_{k \in \mathbb{Z}} e^{\rho|k|} \left(\left| \frac{\partial Q}{\partial \xi_k} \right| + \left| \frac{\partial Q}{\partial \eta_k} \right| \right) \\ &\leq 2M \sum_{k \in \mathbb{Z}} e^{\rho|k|} \sum_{\substack{j_1, \ell_1, \ell_2 \in \mathbb{Z} \\ \mathcal{M}(j_1, k, \ell_1, \ell_2) = \pm 2}} |\xi_{j_1} \eta_{\ell_1} \eta_{\ell_2}| + |\xi_{\ell_1} \xi_{\ell_2} \eta_{j_1}| \\ &\leq 2Me^{2\rho} \sum_{j_1, \ell_1, \ell_2 \in \mathbb{Z}} \left| \xi_{j_1} e^{\rho|j_1|} \eta_{\ell_1} e^{\rho|\ell_1|} \eta_{\ell_2} e^{\rho|\ell_2|} \right| + \left| \xi_{\ell_1} e^{\rho|\ell_1|} \xi_{\ell_2} e^{\rho|\ell_2|} \eta_{j_1} e^{\rho|j_1|} \right| \\ &\leq 4Me^{2\rho} \|(\xi, \eta)\|_\rho^3. \end{aligned}$$

where we used, $\mathcal{M}(j_1, k, \ell_1, \ell_2) = \pm 2 \Rightarrow |k| \leq 2 + |j_1| + |\ell_1| + |\ell_2|$.

Since there are no small divisors in this resonant case, Z_4 and χ are well defined on \mathcal{A}_ρ and by construction Z_4 satisfies (ii). On the other hand, since χ is homogeneous of order 4, for ε sufficiently small, the time one flow generated by χ maps the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ and is close to the identity in the sense of assertion (vii).

Concerning R_5 , by construction it is a Hamiltonian function which is of order at least 5. To obtain assertion (vi) it remains to prove that the vector field X_{R_5} is smooth from $B_\rho(\varepsilon)$ into \mathcal{A}_ρ in such a way we can Taylor expand X_{R_5} at the origin. This is clear for the first term of (2.6), $\{P, \chi\}$. For the second, notice that $\{H, \chi\} = Z_4 - P + \{P, \chi\}$ which is a polynomial on \mathcal{A}_ρ having bounded coefficients⁽²⁾ and the same is true for $Q = \{\{H, \chi\}, \chi\}$. Therefore, in view of the previous paragraph, X_Q is smooth. Now, since for ε small enough Φ_χ^t maps smoothly the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ for all $0 \leq t \leq 1$, we conclude that $\int_0^1 (1-t) \{\{H, \chi\}, \chi\} \circ \Phi_\chi^t dt$ has a smooth vector field.

We are now interested in describing more explicitly the terms appearing in the expression (2.8) on base of the number of times that $\xi_{\pm 1}$ or $\eta_{\pm 1}$ appear as

2. Notice that this is not true for H_0 .

a factor. Let us define the resonant set

$$(2.10) \quad \mathcal{R} = \{(j_1, j_2, j_3, j_4) \in \mathbb{Z}^4 \mid j_1 + j_2 - j_3 - j_4 = \pm 2 \text{ and } j_1^2 + j_2^2 - j_3^2 - j_4^2 = 0\}$$

in such a way (2.8) reads

$$(2.11) \quad Z_4 = \sum_{(j_1, j_2, j_3, j_4) \in \mathcal{R}} \xi_{j_1} \xi_{j_2} \eta_{j_3} \eta_{j_4}.$$

Note that it is enough to deal with the case $j_1 + j_2 - j_3 - j_4 = 2$ because the other case $j_1 + j_2 - j_3 - j_4 = -2$ can be obtained by changing the signs of j_1, j_2, j_3, j_4 . Thus let us study two cases: $j_1 = 1$ or $j_1 = -1$.

In the first case we have $j_2 = j_3 + j_4 + 1$ and therefore the quadratic condition implies $(1 + j_3)(1 + j_4) = 0$. Thus we conclude that $(j_1, j_2, j_3, j_4) = (1, p, -1, p)$ or $(j_1, j_2, j_3, j_4) = (1, p, p, -1)$. Thus when $j_1 = 1$, the only terms appearing in the expression for Z_4 in Eq. (2.11) must have the form $\xi_1, \xi_p, \eta_{-1}\eta_p$ or $\xi_1, \xi_p, \eta_p, \eta_{-1}$.

In the second case, $j = -1$, we have $j_2 = j_3 + j_4 + 3$ which in turn implies $5 + 3(j_3 + j_4) + j_3 j_4 = 0$. By looking at the graph of the function $f(x) = -\frac{5+3x}{3+x}$ we find that the only acceptable cases appearing in Eq. (2.11) are $(j_1, j_2, j_3, j_4) = (-1, 1, -1, -1), (-1, 2, -2, 1), (-1, 2, 1, -2), (-1, -8, -4, -7), (-1, -8, -7, -4)$, or $(-1, -7, -5, -5)$.

Noting that the resonant set \mathcal{R} is invariant under permutations of j_1 with j_2 or j_3 with j_4 we obtain that

$$\begin{aligned} Z_4 = & \left(4 \sum_{p \in \mathbb{Z}} \xi_p \eta_p - 2(\xi_1 \eta_1 + \xi_{-1} \eta_{-1}) \right) (\xi_1 \eta_{-1} + \xi_{-1} \eta_1) \\ & + 4(\xi_2 \xi_{-1} \eta_{-2} \eta_1 + \xi_{-2} \xi_1 \eta_2 \eta_{-1}) \\ & + 4(\xi_{-1} \xi_{-8} \eta_{-7} \eta_{-4} + \xi_1 \xi_8 \eta_7 \eta_4) + 2(\xi_{-1} \xi_{-7} \eta_{-5} \eta_{-5} + \xi_1 \xi_7 \eta_5 \eta_5) + \tilde{Z}_4 \end{aligned}$$

where \tilde{Z}_4 is equal to the sum of terms of the form $\xi_{j_1} \xi_{j_2} \eta_{j_3} \eta_{j_4}$ with $(j_1, j_2, j_3, j_4) \in \mathcal{R}$ and satisfying the condition $|j_k| \neq 1$, for all $k = 1, 2, 3, 4$. \square

3. Dynamical consequences

We denote by $I_p = \xi_p \eta_p$, $p \in \mathbb{Z}$, the actions, by $J_p = I_p + I_{-p}$ for $p \in \mathbb{N} \setminus \{0\}$ and $J_0 = I_0$, the generalized actions and by $J = \sum_{p \in \mathbb{Z}} I_p$. Notice that, when the initial condition (ξ^0, η^0) of the Hamiltonian system (2.1) satisfies $\eta^0 = \bar{\xi}^0$ (and this is actually the case when ξ^0 and η^0 are the sequences of Fourier coefficients of respectively $\psi_0, \bar{\psi}_0$), this reality property is conserved i.e. $\eta(t) = \bar{\xi}(t)$ for all t . As a consequence all the quantities I_p, J_p and J are real and positive.

To begin with we establish that apart from I_1 and I_{-1} the others actions are almost constant:

Lemma 3.1. — *Let $\psi(t, \cdot) = \sum_{k \in \mathbb{Z}} \xi_k(t) e^{ikx}$ be the solution of (1.3) then for ε small enough and $|t| \leq \varepsilon^{-\frac{9}{4}}$,*

$$|J(t) - J(0)| \leq \varepsilon^{5/2} \text{ and } J_p(t) \leq \varepsilon^3 \text{ for } p \in \mathbb{N} \setminus \{1\}.$$

As a consequence

$$I_1(t), I_{-1}(t) \leq 4\varepsilon^2 \text{ while } I_p(t) \leq \varepsilon^3 \text{ for } p \in \mathbb{N} \setminus \{1\}.$$

Proof. — Using Proposition 2.1 we have for all $p \in \mathbb{N}$

$$\dot{J}_p = \{J_p, H\} = \{J_p, Z_{4,3}\} + \{J_p, R\}$$

since by elementary calculations $\{J_p, Z_{4,1}\} = \{J_p, Z_{4,2}\} = 0$. In particular

$$\dot{J} = \{J, H\} = \{J, Z_{4,3}\} + \{J, R\}.$$

Let T_ε be the maximal time such that for all $|t| \leq T_\varepsilon$

$$|J(t) - J(0)| \leq \varepsilon^{5/2} \text{ and } J_p(t) \leq \varepsilon^3, \text{ for } p \in \mathbb{N} \setminus \{1\}.$$

Since $J(0) = \varepsilon^2$, we get for ε small enough and $|t| \leq T_\varepsilon$

$$|\xi_1(t)|, |\xi_{-1}(t)|, |\eta_1(t)|, |\eta_{-1}(t)| \leq 2\varepsilon \text{ while } |\xi_p(t)|, |\eta_p(t)| \leq \varepsilon^{3/2} \text{ for } p \neq 1.$$

Therefore, from the definition of $Z_{4,3}$ and R we deduce that for $|t| \leq T_\varepsilon$

$$\begin{aligned} \{J_p, R\} &= O(\varepsilon^{\frac{11}{2}}), \quad \{J_p, Z_{4,3}\} = O(\varepsilon^{\frac{11}{2}}) \text{ for } p \neq 1, \\ \{J, R\} &= O(\varepsilon^5), \quad \{J, Z_{4,3}\} = O(\varepsilon^{\frac{11}{2}}), \end{aligned}$$

and thus there exists $C > 0$ such that for any $|t| \leq T_\varepsilon$

$$|\dot{J}(t)| \leq C\varepsilon^5 \text{ and } |\dot{J}_p| \leq C\varepsilon^{\frac{11}{2}} \text{ for } p \neq 1.$$

Taking into account that $J_p(0) = 0$ for $p \neq 1$, we obtain

$$|J(t) - J(0)| \leq C|t|\varepsilon^5, \text{ and } |J_p(t)| \leq C|t|\varepsilon^{\frac{11}{2}} \text{ for } p \neq 1$$

for all $|t| \leq T_\varepsilon$ and we conclude by a classical bootstrap argument that $T_\varepsilon \geq C^{-1}\varepsilon^{-5/2} \geq \varepsilon^{-9/4}$ for ε small enough. \square

In order to prove theorem 1.1 let us define some quadratic Hamiltonian functions ($p \in \mathbb{Z}$):

$$\begin{aligned} M_p &:= \xi_p \eta_p - \xi_{-p} \eta_{-p}, & J_p &:= \xi_p \eta_p + \xi_{-p} \eta_{-p}, \\ L_p &:= \imath(\xi_p \eta_{-p} - \xi_{-p} \eta_p), & K_p &:= \xi_p \eta_{-p} + \xi_{-p} \eta_p. \end{aligned}$$

One computes

$$\begin{aligned}\dot{M}_1 &= \{M_1, H\} = \{M_1, Z_{4,1} + Z_{4,2}\} + \{M_1, Z_{4,3} + R\} \\ &= 2JL_1 + L_1K_2 - K_1L_2 + \{M_1, Z_{4,3} + R\}\end{aligned}$$

and using Lemma 3.1 we get that for $|t| \leq \varepsilon^{-9/4}$,

$$\dot{M}_1 = 2J(0)L_1 + O(\varepsilon^{9/2})$$

and in the same way we verify

$$\dot{L}_1 = -2J(0)M_1 + O(\varepsilon^{9/2}).$$

We can now compute the solution of the associated linear ODE

$$\begin{cases} \dot{M}_1 = 2J(0)L_1 \\ \dot{L}_1 = -2J(0)M_1 \end{cases}$$

to conclude that for $t \leq C\varepsilon^{-9/4}$

$$M_1(t) = M_1(0) \cos 2J(0)t + L_1(0) \sin 2J(0)t + O(\varepsilon^{9/4}).$$

In Theorem 1.1 we have chosen $\psi_0 = \varepsilon(\cos x + \sin x)$ which corresponds to $\xi_1 = \eta_{-1} = \frac{1-i}{2}\varepsilon$ and $\eta_1 = \xi_{-1} = \frac{1+i}{2}\varepsilon$. Therefore $J(0) = J_1(0) = \varepsilon^2$, $M_1(0) = 0$ and $L_1(0) = \varepsilon^2$ which leads to the desired result.

We finally remark that choosing the minus sign in front of the nonlinearity will lead to the following linear system

$$\begin{cases} \dot{M}_1 = -2J(0)L_1 \\ \dot{L}_1 = 2J(0)M_1 \end{cases}$$

which again gives the desired result.

4. Generalizations and comments

- The same result remains true when we add a higher order term to the nonlinearity, i.e. considering the equation

$$i\psi_t = -\psi_{xx} \pm 2 \cos 2x |\psi|^2 \psi + O(|\psi|^4), \quad x \in S^1, \quad t \in \mathbb{R}.$$

- We can prove a similar result when changing the nonlinearity in such a way we still privilege the modes 1 and -1. The game is to conserve an effective Hamiltonian at order 4 (see Proposition 2.1). For instance $2 \cos 2x$ can be replaced by $a \cos 2x + b \sin 2x$ but not by $\cos 4x$ which generates an effective Hamiltonian only at order 6. We can also choose to privilege another couple of modes p and $-p$ choosing $a \cos 2px + b \sin 2px$. In that case we have also to adapt the initial datum.

- If we choose a non linearity that does not depend on x , for instance the standard cubic nonlinearity $g = \pm|\psi|^2\psi$, then we can prove that there is no beating effect between any modes for $|t| \leq \varepsilon^{-3}$ since Z_4 only depends on the actions in that case (independently of the sign in front of the nonlinearity).
- We can also change the initial datum. Remark that if you chose $\psi_0 = \varepsilon \cos x$ or $\psi_0 = \varepsilon \sin x$, no beating effect appears at order 4 since in both cases $M_1(0) = L_1(0) = 0$.
- The beating frequency, namely $2J(0)$, depends on all the modes initially excited. For instance if $\psi_0 = \varepsilon(\cos x + \sin x) + \varepsilon^2 \cos qx$ ($q \neq 1$) then we can still prove that there is no energy exchanges between the mode p , for $|p| \neq 1$, and modes 1 and -1 , that there is the same beating effect between modes 1 and -1 , nevertheless the beating frequency is slightly changed: $2J(0) = 2\varepsilon^2 + \varepsilon^4$.
- As stated in the introduction, when adding a linear potential –a multiplicative one or a convolution one– we can choose the potential in order to avoid resonances between the different blocks of modes p and $-p$ (see [BG06] for multiplicative potentials or [Gré07] for convolution potentials). In that case the same result can be proved and actually in an easier way since we avoid exchanges between the blocks for arbitrary long time.

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